

Exam Category Theory

19 January 2026

- The exam will last from 10:00-13:00.
- There are 4 exercises.
- You can ask for clarification if you do not understand an exercise, but we will not provide hints.
- Good luck!

Exercise 1 Are the following statements true or false? You do not need to justify your answer.

- (a) (5 points) If $\mathcal{C}_0 \simeq \mathcal{D}_0$ and $\mathcal{C}_1 \simeq \mathcal{D}_1$, then $\mathcal{C}_0 \times \mathcal{C}_1 \simeq \mathcal{D}_0 \times \mathcal{D}_1$.
- (b) (5 points) If \mathcal{C} and \mathcal{D} are categories and \mathcal{C} is complete, then so is $[\mathcal{C}, \mathcal{D}]$.
- (c) (5 points) For any small category \mathcal{C} , the Yoneda embedding $\mathfrak{y}: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ preserves epis.
- (d) (5 points) For any small category \mathcal{C} , the Yoneda embedding $\mathfrak{y}: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ preserves monos.

Exercise 2 In this exercise it suffices to give the constructions: you are not asked to verify functoriality.

We will call a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ a *cofibration* if it is injective on objects; and a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ a *trivial fibration* if it is fully faithful and surjective in objects.

- (a) (10 points) Show that each functor $F: \mathcal{C} \rightarrow \mathcal{D}$ can be written as $F = H \circ G$ where $G: \mathcal{C} \rightarrow \mathcal{E}$ is a cofibration and $H: \mathcal{E} \rightarrow \mathcal{D}$ is a trivial fibration.

Hint: Put $\mathcal{E}_0 = \mathcal{C}_0 + \mathcal{D}_0$.

- (b) (10 points) Imagine that we are given a solid commutative square of the form

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{M} & \mathcal{E} \\
 F \downarrow & \swarrow L & \downarrow G \\
 \mathcal{D} & \xrightarrow{N} & \mathcal{F}
 \end{array}$$

in **Cat** in which F is a cofibration and G a trivial fibration. Show that there is a functor $L: \mathcal{D} \rightarrow \mathcal{E}$ making both resulting triangles commute.

Exercise 3 If \mathcal{C} is a category, let us write $T(\mathcal{C})$ for the category of (not necessarily commutative) triangles in \mathcal{C} . More precisely, $T(\mathcal{C})$ is the following category:

- Objects are 6-tuples $(A_0, A_1, A_2, f_0, f_1, f_2)$ where the A_i are objects in \mathcal{C} and $f_0: A_0 \rightarrow A_1$, $f_1: A_1 \rightarrow A_2$ and $f_2: A_0 \rightarrow A_2$ are morphisms in \mathcal{C} .
- Morphisms $(A_0, A_1, A_2, f_0, f_1, f_2) \rightarrow (B_0, B_1, B_2, g_0, g_1, g_2)$ are triples $(\varphi_i: A_i \rightarrow B_i: i \in \{0, 1, 2\})$ of maps in \mathcal{C} such that $\varphi_1 \circ f_0 = g_0 \circ \varphi_0$, $\varphi_2 \circ f_1 = g_1 \circ \varphi_1$ and $\varphi_2 \circ f_2 = g_2 \circ \varphi_0$.
- Composition and identities are componentwise as in \mathcal{C} .

(You are not asked to prove that this defines a category.) In addition, let $CT(\mathcal{C})$ be the full subcategory of $T(\mathcal{C})$ on the commutative triangles (that is, it has as objects those 6-tuples $(A_0, A_1, A_2, f_0, f_1, f_2)$ for which $f_2 = f_1 \circ f_0$ and the same morphisms as $T(\mathcal{C})$); we will write $I: CT(\mathcal{C}) \rightarrow T(\mathcal{C})$ for the inclusion functor.

- (10 points) Show that both $CT(\mathbf{Sets})$ and $T(\mathbf{Sets})$ are isomorphic to presheaf categories, and use this to prove that I has both adjoints if $\mathcal{C} = \mathbf{Sets}$.
- (10 points) Show that I has a left adjoint if \mathcal{C} has ~~pushouts~~. *coeqs.*
- (10 points) Show that I has a right adjoint if \mathcal{C} has ~~pullbacks~~. *eqs.*

Exercise 4 Let \mathcal{F} be the full subcategory of the category of sets on finite sets of the form $\bar{n} = \{1, \dots, n\}$ with $n \in \mathbb{N}$ (this includes the empty set for $n = 0$). Let us say that a sieve S on \bar{n} in \mathcal{F} is covering if for each $i \in \bar{n}$ there is an arrow $\alpha: \bar{m} \rightarrow \bar{n} \in S$ and a $j \in \bar{m}$ with $\alpha(j) = i$.

- (10 points) Show that this defines a Grothendieck topology Cov on \mathcal{F} .
- (10 points) Let X be a sheaf on the site $(\mathcal{F}, \text{Cov})$. Show that $X(\bar{n}) \cong X(\bar{1})^n$ for each $n \in \mathbb{N}$.
- (10 points) Show that $\text{Sh}(\mathcal{F}, \text{Cov}) \simeq \mathbf{Sets}$.